On sumsets in $\mathbb{F}_2^n$

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Abstract. Let $\mathbb{F}_2$ be the finite field of two elements, $\mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. For sets A, $B \subseteq \mathbb{F}_2^n$, their sumset is defined as the set of all pairwise sums $a+b$ with $a \in A$, $b \in B$.

Ben Green and Terence Tao proved that, let $K \geq 1$, if A, $B \subseteq \mathbb{F}_2^n$ and $|A+B| \leq K|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$, then there exists a subspace $H \subseteq \mathbb{F}_2^n$ with

$$|H| \gg \exp(-O(\sqrt{K}\log K))|A|$$

and x, $y \in \mathbb{F}_2^n$ such that

$$|A \cap (x+H)|^{\frac{1}{2}}|B \cap (y+H)|^{\frac{1}{2}} \geq \frac{1}{2K}|H|.$$

In this note, we shall use the method of Green and Tao with some modification to prove that if

$$|H| \gg \exp(-O(\sqrt{K}))|A|,$$

then the above conclusion still holds true.

1. Introduction

Let $\mathbb{F}_2$ be the finite field of two elements, $\mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. For sets A, $B \subseteq \mathbb{F}_2^n$, their sumset $A+B$ is defined as

$$A+B := \{a+b: \ a \in A, \, b \in B\}.$$

In 1999, Ruzsa[4] proved the following theorem.

Theorem 1(Ruzsa). Let $K \geq 1$ be an integer, and suppose that set $A \subseteq \mathbb{F}_2^n$ with $|A+A| \leq K|A|$. Then A is contained in a subspace $H \subseteq \mathbb{F}_2^n$ with $|H| \leq F(K)|A|$, where $F(K) = K^2 2^{K^4}$.

This result was improved by Sanders[5] to $F(K) = 2^{O(K^{\frac{3}{2}}\log K)}$ in 2008 and then improved by Green and Tao[2] to $F(K) = 2^{2K+O(\sqrt{K}\log K)}$ in 2009. The bound $F(K) = 2^{2K+O(\sqrt{K}\log K)}$ is almost best possible.

If we do not require that the subspace H contains the set A completely but contains a part of A, then related bounds can be further improved.

The following theorem was given in [1] and some explanations on it could be found in the introduction of [3].

Theorem 2. Suppose that $K \geq 1$ and that $A \subseteq \mathbb{F}_2^n$ with $|A + A| \leq K|A|$. Then there is a subspace $H \subseteq \mathbb{F}_2^n$ with $|H| \ll K^{O(1)}|A|$ such that

$$|A \cap H| \gg \exp(-K^{O(1)})|A|.$$

If we permit to replace the subspace H by translates of it, then better bounds could be obtained. In 2009, Green and Tao[3] obtained the following result.

Theorem 3(Green-Tao). Let $K \geq 1$, if $A,\, B \subseteq \mathbb{F}_2^n$ and $|A + B| \leq K|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$, then there exists a subspace $H \subseteq \mathbb{F}_2^n$ with

$$|H| \gg \exp(-O(\sqrt{K}\log K))|A|$$

and $x,\, y \in \mathbb{F}_2^n$ such that

$$|A \cap (x + H)|^{\frac{1}{2}}|B \cap (y + H)|^{\frac{1}{2}} \geq \frac{1}{2K}|H|.$$

In this note, we shall use the method of Green and Tao with some modification to prove the following theorem.

Theorem 4. Let $K \geq 1$, if $A,\, B \subseteq \mathbb{F}_2^n$ and $|A+B| \leq K|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$, then there exists a subspace $H \subseteq \mathbb{F}_2^n$ with

$$|H| \gg \exp(-O(\sqrt{K}))|A|$$

and $x,\, y \in \mathbb{F}_2^n$ such that

$$|A \cap (x + H)|^{\frac{1}{2}}|B \cap (y + H)|^{\frac{1}{2}} \geq \frac{1}{2K}|H|.$$

2. Definitions

In this section we shall introduce some definitions given in [3].

Definition 1(normalized energy). For non-empty sets $A_1,\, A_2,\, A_3,\, A_4 \subseteq \mathbb{F}_2^n$, define the normalized energy

$$\omega(A_1,\, A_2,\, A_3,\, A_4) := \frac{1}{(|A_1||A_2||A_3||A_4|)^{\frac{3}{4}}}\,|\{(a_1,\, a_2,\, a_3,\, a_4) \in A_1 \times A_2 \times A_3 \times A_4 :\ a_1 + a_2 + a_3 + a_4 = 0\}|.$$

It was shown in [3] that

$$0 \le \omega(A_1,\, A_2,\, A_3,\, A_4) \le 1. \tag{1}$$

Definition 2(Fourier transform). For $f : \mathbb{F}_2^n \longrightarrow \mathbb{R}$, define the Fourier transform $\hat{f} : \mathbb{F}_2^n \longrightarrow \mathbb{R}$ by

$$\hat{f}(\xi) := \frac{1}{2^n} \sum_{x \in \mathbb{F}_2^n} f(x)(-1)^{\xi \cdot x},$$

where

$$\xi \cdot x = (\xi_1,\, \cdots,\, \xi_n) \cdot (x_1,\, \cdots,\, x_n) = \xi_1 x_1 + \cdots + \xi_n x_n.$$

Definition 3(spectrum). If $A \subseteq \mathbb{F}_2^n$ is non-empty and $0 < \alpha \le 1$, define the $\alpha-$spectrum

$$\mathrm{Spec}_\alpha(A) := \{\xi \in \mathbb{F}_2^n :\ |\hat{\mathbf{1}}_A(\xi)| \ge \alpha \frac{|A|}{2^n}\},$$

where $\mathbf{1}_A(x)$ is the indicator function of set A.

Definition 4(coherently flat quadruples). Suppose that $A_1,\, A_2,\, A_3,\, A_4 \subseteq \mathbb{F}_2^n$ are non-empty and $\delta \in (0,\, \frac{1}{2})$ is a small parameter. If for each $\xi \in \mathbb{F}_2^n$, one of the following conditions is satisfied:

1) $\xi \in \mathrm{Spec}_{\frac{9}{10}}(A_i)$ for all $i = 1,\, 2,\, 3,\, 4$;

2) $\xi \notin \mathrm{Spec}_\delta(A_i)$ for all $i = 1,\, 2,\, 3,\, 4$,

we say that the quadruple $(A_1,\, A_2,\, A_3,\, A_4)$ is coherently $\delta-$ flat.

3. The proof of Theorem 4

Lemma 1. Let $J \ge 1$. Suppose that $(A_1,\, A_2,\, A_3,\, A_4)$ is a coherently $\frac{1}{\sqrt{2J}}-$flat quadruple, the normalized energy of which satisfies

$$\omega(A_1,\, A_2,\, A_3,\, A_4) \ge \frac{1}{J}.$$

Then there is a subspace $H \subseteq \mathbb{F}_2^n$ with $x_1,\, x_2,\, x_3,\, x_4 \in \mathbb{F}_2^n$ such that

$$H \ge \frac{4}{5}\,(|A_1||A_2||A_3||A_4|)^{\frac{1}{4}} \tag{2}$$

and

$$\prod_{i=1}^{4} |A_i \cap (x_i + H)|^{\frac{1}{4}} \ge \frac{1}{2J}\,|H|. \tag{3}$$

This is Proposition 2.4 in [3].

Let

$$\mathrm{Dbl}(A,\, B) := \frac{|A+B|}{|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}}.$$

Since

$$|A+B| \geq \max(|A|,\, |B|),$$

we have

$$\mathrm{Dbl}(A,\, B) \geq 1. \tag{4}$$

Lemma 2. Suppose that $A,\, B \subseteq \mathbb{F}_2^n$ are non-empty and that for $J \geq 1$,

$$\mathrm{Dbl}(A,\, B) \leq J.$$

If $(A,\, B,\, A,\, B)$ is not coherently $\frac{1}{\sqrt{2J}}$−flat, then there are $A' \subseteq A$, $B' \subseteq B$ such that

$$|A'| \geq \frac{1}{20}\,|A|, \qquad\qquad |B'| \geq \frac{1}{20}\,|B| \tag{5}$$

and

$$\mathrm{Dbl}(A',\, B') \leq \frac{J}{1+\frac{1}{100\sqrt{J}}}. \tag{6}$$

Proof. By the supposition, there is $\xi \in \mathbb{F}_2^n$ such that

$$\xi \notin \mathrm{Spec}_{\frac{9}{10}}(A) \cap \mathrm{Spec}_{\frac{9}{10}}(B) \tag{7}$$

and

$$\xi \in \mathrm{Spec}_{\frac{1}{\sqrt{2J}}}(A) \cup \mathrm{Spec}_{\frac{1}{\sqrt{2J}}}(B). \tag{8}$$

By (7), $\xi \neq 0$. Write

$$A_0 := \{x \in A:\ x\cdot\xi = 0\}, \qquad A_1 := \{x \in A:\ x\cdot\xi = 1\},$$
$$B_0 := \{x \in B:\ x\cdot\xi = 0\}, \qquad B_1 := \{x \in B:\ x\cdot\xi = 1\}.$$

If $|A_0| \geq \frac{1}{2}|A|$, we write $\alpha := \frac{|A_0|}{|A|}$. Otherwise, $|A_0| < \frac{1}{2}|A| \Rightarrow |A_1| = |A| - |A_0| \geq |A| - \frac{1}{2}|A| = \frac{1}{2}|A|$. Then we write $\alpha := \frac{|A_1|}{|A|}$. Without loss of generality, we can suppose that $|A_0| \geq \frac{1}{2}|A|$ and write

$$\alpha := \frac{|A_0|}{|A|}.$$

Similarly, we can also suppose that $|B_0| \geq \frac{1}{2}|B|$ and write

$$\beta := \frac{|B_0|}{|B|}.$$

We have

$$\alpha \geq \frac{1}{2}, \qquad\qquad \beta \geq \frac{1}{2}. \tag{9}$$

By

$$\begin{aligned}
|\hat{\mathbf{1}}_A(\xi)| &= \Big|\frac{1}{2^n}\sum_{x\in A}(-1)^{x\cdot\xi}\Big| \\
&= \Big|\frac{1}{2^n}\Big(\sum_{x\in A_0}(-1)^{x\cdot\xi} + \sum_{x\in A_1}(-1)^{x\cdot\xi}\Big)\Big| \\
&= \Big|\frac{1}{2^n}\left(|A_0| - |A_1|\right)\Big| \\
&= \Big|\frac{1}{2^n}\left(2|A_0| - |A|\right)\Big| \\
&= (2\alpha - 1)\cdot\frac{|A|}{2^n}
\end{aligned}$$

and

$$|\hat{\mathbf{1}}_B(\xi)| = (2\beta - 1)\cdot\frac{|B|}{2^n},$$

we know that the condition (7) is equivalent to

$$2\alpha - 1 < \frac{9}{10} \qquad \text{or} \qquad 2\beta - 1 < \frac{9}{10}, \tag{10}$$

and the condition (8) is equivalent to

$$2\alpha - 1 \geq \frac{1}{\sqrt{2J}} \qquad \text{or} \qquad 2\beta - 1 \geq \frac{1}{\sqrt{2J}}. \tag{11}$$

Without loss of generality, we suppose that

$$\beta \geq \alpha \tag{12}$$

and consider

$$|B_0 + A_0| + |B_0 + A_1|.$$

If $\beta < \alpha$, we shall consider $|A_0 + B_0| + |A_0 + B_1|$.

It is easy to see that sets $B_0 + A_0$ and $B_0 + A_1$ are disjoint. Hence,

$$|B_0 + A_0| + |B_0 + A_1| \leq |B + A| \leq J|B|^{\frac{1}{2}}|A|^{\frac{1}{2}}$$

or

$$\frac{|B_0+A_0|}{|B_0|^{\frac{1}{2}}|A_0|^{\frac{1}{2}}}\cdot\frac{|B_0|^{\frac{1}{2}}|A_0|^{\frac{1}{2}}}{|B|^{\frac{1}{2}}|A|^{\frac{1}{2}}}+\frac{|B_0+A_1|}{|B_0|^{\frac{1}{2}}|A_1|^{\frac{1}{2}}}\cdot\frac{|B_0|^{\frac{1}{2}}|A_1|^{\frac{1}{2}}}{|B|^{\frac{1}{2}}|A|^{\frac{1}{2}}}\le J. \tag{13}$$

Let

$$\Psi := \min\Big(\frac{|B_0+A_0|}{|B_0|^{\frac{1}{2}}|A_0|^{\frac{1}{2}}}, \frac{|B_0+A_1|}{|B_0|^{\frac{1}{2}}|A_1|^{\frac{1}{2}}}\Big).$$

It follows from (13) that

$$\Psi(\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}})\le J. \tag{14}$$

Under the supposition (12), the condition (10) is equivalent to

$$\alpha<\frac{19}{20}, \tag{15}$$

and the condition (11) is equivalent to

$$\beta\ge\frac{1}{2}+\frac{1}{2\sqrt{2J}}. \tag{16}$$

We shall discuss in the following two cases.

Case 1. $\frac{1}{2}+\frac{1}{2\sqrt{2J}}\le\alpha<\frac{19}{20}$.

By (12),

$$\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\ge\alpha+\alpha^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}.$$

The discussion in [3] yields that

$$\begin{aligned}&\alpha+\alpha^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\ge\alpha+2\alpha(1-\alpha)\\ &=1+(2\alpha-1)(1-\alpha)\ge1+\frac{1}{20\sqrt{2J}}.\end{aligned}$$

Hence,

$$\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\ge1+\frac{1}{20\sqrt{2J}}.$$

Case 2. $\frac{1}{2}\le\alpha<\frac{1}{2}+\frac{1}{2\sqrt{2J}}$.

It follows from (16) that

$$\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\ge\Big(\frac{1}{2}+\frac{1}{2\sqrt{2J}}\Big)^{\frac{1}{2}}(\alpha^{\frac{1}{2}}+(1-\alpha)^{\frac{1}{2}}).$$

Let

$$f(\alpha)=\alpha^{\frac{1}{2}}+(1-\alpha)^{\frac{1}{2}}.$$

Since

$$f'(\alpha) = \frac{1}{2\sqrt{\alpha}} - \frac{1}{2\sqrt{1-\alpha}} \le 0,$$

the function $f(\alpha)$ is decreasing monotonically. Thus,

$$\alpha^{\frac{1}{2}} + (1-\alpha)^{\frac{1}{2}} \ge \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)^{\frac{1}{2}} + \Big(1 - \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)\Big)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned}
&\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}} \\
&\ge \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)^{\frac{1}{2}}\Big(\Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)^{\frac{1}{2}} + \Big(1 - \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)\Big)^{\frac{1}{2}}\Big) \\
&= \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big) + \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)^{\frac{1}{2}}\Big(1 - \Big(\frac{1}{2} + \frac{1}{2\sqrt{2J}}\Big)\Big)^{\frac{1}{2}}
\end{aligned}$$

which is the value of function $\alpha + \alpha^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}$ at $\alpha = \frac{1}{2} + \frac{1}{2\sqrt{2J}}$. By the discussion in Case 1, we have

$$\beta^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}} \ge 1 + \frac{1}{20\sqrt{2J}}.$$

Combining the above two cases, we get

$$\Psi \le \frac{J}{1 + \frac{1}{20\sqrt{2J}}} \le \frac{J}{1 + \frac{1}{100\sqrt{J}}}.$$

Take $B' = B_0$, $A' = A_0$ or A_1 such that

$$\Psi = \mathrm{Dbl}(A',\, B').$$

Then

$$\mathrm{Dbl}(A',\, B') \le \frac{J}{1 + \frac{1}{100\sqrt{J}}}.$$

Since

$$|A_0| \ge \frac{1}{2}|A|, \qquad |A_1| = |A| - |A_0| \ge |A| - \frac{19}{20}|A| = \frac{1}{20}|A|,$$

we have

$$|A'| \ge \frac{1}{20}|A|.$$

We also have

$$|B'| \ge \frac{1}{20}|B|.$$

So far the proof of Lemma 2 is finished.

Lemma 3. Suppose that A, $B \subseteq \mathbb{F}_2^n$ are non-empty and that for $K \geq 1$,

$$\mathrm{Dbl}(A,\, B) \leq K.$$

Then there are $A' \subseteq A$, $B' \subseteq B$ with

$$|A'| \gg \exp(-O(\sqrt{K}))|A|, \qquad |B'| \gg \exp(-O(\sqrt{K}))|B| \tag{17}$$

such that for some $J (1 \leq J \leq K)$,

$$\mathrm{Dbl}(A',\, B') \leq J \tag{18}$$

and $(A',\, B',\, A',\, B')$ is coherently $\frac{1}{\sqrt{2J}}$−flat.

Proof. Take $K_1 = K$. If $(A,\, B,\, A,\, B)$ is coherently $\frac{1}{\sqrt{2K}}$−flat, then the conclusion holds true.

If $(A,\, B,\, A,\, B)$ is not coherently $\frac{1}{\sqrt{2K}}$−flat, Lemma 2 produces that there are $A'' \subseteq A$, $B'' \subseteq B$ with

$$|A''| \geq \frac{1}{20}|A|, \qquad |B''| \geq \frac{1}{20}|B|$$

such that

$$\mathrm{Dbl}(A'',\, B'') \leq \frac{K_1}{1 + \frac{1}{100\sqrt{K_1}}}.$$

Then take

$$K_2 = \frac{K_1}{1 + \frac{1}{100\sqrt{K_1}}},$$

and for A'', B'' and K_2, repeat the above process.

Since $\mathrm{Dbl} \geq 1$, this process has to stop after finite steps. We get a sequence $K_1 = K,\, K_2,\, \cdots,\, K_m = J$ with

$$K_{i+1} = \frac{K_i}{1 + \frac{1}{100\sqrt{K_i}}}, \qquad i = 1,\, 2,\, \cdots,\, m-1$$

and $A' \subseteq A$, $B' \subseteq B$ with

$$|A'| \gg \frac{1}{(20)^m}|A|, \qquad |B'| \gg \frac{1}{(20)^m}|B|$$

such that

$$\mathrm{Dbl}(A',\, B') \leq J$$

and $(A',\,B',\,A',\,B')$ is coherently $\frac{1}{\sqrt{2J}}$−flat.

We distribute K_i into intervals

$$\Big(\frac{K}{e^{r+1}},\,\frac{K}{e^{r}}\Big],\,\Big(\frac{K}{e^{r}},\,\frac{K}{e^{r-1}}\Big],\,\cdots,\,\Big(\frac{K}{e^{2}},\,\frac{K}{e}\Big],\,\Big(\frac{K}{e},\,K\Big],\quad r=[\log K].$$

For the given interval $(\frac{K}{e^{s+1}},\,\frac{K}{e^{s}}](0\le s\le r)$, if K_l and $K_{l+j}(j\ge 1)\in(\frac{K}{e^{s+1}},\,\frac{K}{e^{s}}]$, we have

$$\begin{aligned}\frac{K}{e^{s+1}}\le K_{l+j}&=\frac{K_{l+j-1}}{1+\frac{1}{100\sqrt{K_{l+j-1}}}}\le\frac{K_{l+j-1}}{1+\frac{1}{100\sqrt{\frac{K}{e^s}}}}\le\cdots\\&\le\frac{K_l}{\Big(1+\frac{1}{100\sqrt{\frac{K}{e^s}}}\Big)^j}\le\frac{K}{e^s}\cdot\frac{1}{\Big(1+\frac{1}{100\sqrt{\frac{K}{e^s}}}\Big)^j}.\end{aligned}$$

Thus

$$\begin{aligned}\Big(1+\frac{1}{100\sqrt{\frac{K}{e^s}}}\Big)^j&\le e,\\ j\cdot\frac{1}{\sqrt{\frac{K}{e^s}}}\ll j\log\Big(1+\frac{1}{100\sqrt{\frac{K}{e^s}}}\Big)&\le 1,\\ j\ll\sqrt{\frac{K}{e^s}}.\end{aligned}$$

Hence, the number of K_i dropping into the interval $(\frac{K}{e^{s+1}},\,\frac{K}{e^{s}}]$ is $\ll\sqrt{\frac{K}{e^s}}$. For the total number of K_i, we have

$$\begin{aligned}m&\ll\sqrt{K}+\sqrt{\frac{K}{e}}+\sqrt{\frac{K}{e^2}}+\cdots+\sqrt{\frac{K}{e^r}}\\&\le\sqrt{K}\Big(1+\frac{1}{\sqrt{e}}+\frac{1}{(\sqrt{e})^2}+\frac{1}{(\sqrt{e})^3}+\cdots\Big)\\&\ll\sqrt{K}.\end{aligned}$$

Therefore

$$|A'|\gg\exp(-O(\sqrt{K}))|A|,\qquad|B'|\gg\exp(-O(\sqrt{K}))|B|.$$

So far the proof of Lemma 3 is finished.

The proof of Theorem 4. We take A', B' in Lemma 3 with required properties. It is shown in [3] that

$$\omega(A',\,B',\,A',\,B')\ge\frac{1}{\mathrm{Dbl}(A',\,B')}\ge\frac{1}{J}.$$

Lemma 1 claims that there is a subspace $H \subseteq \mathbb{F}_2^n$ with $x_1, x_2, x_3, x_4 \in \mathbb{F}_2^n$ such that

$$H \geq \frac{4}{5}\,|A'|^{\frac{1}{2}}|B'|^{\frac{1}{2}} \gg \exp(-O(\sqrt{K}))|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$$

and

$$\begin{aligned}&|A \cap (x_1 + H)|^{\frac{1}{4}}|B \cap (x_2 + H)|^{\frac{1}{4}}|A \cap (x_3 + H)|^{\frac{1}{4}}|B \cap (x_4 + H)|^{\frac{1}{4}} \\ &\geq \frac{1}{2J}\,|H| \geq \frac{1}{2K}\,|H|.\end{aligned}$$

Since

$$|A| \leq |A + B| \leq K|A|^{\frac{1}{2}}|B|^{\frac{1}{2}},$$

we have

$$K^{-2}|A| \leq |B|,$$

hence

$$H \gg \exp(-O(\sqrt{K}))|A|.$$

Without loss of generality, we can suppose that

$$|A \cap (x_1 + H)| \geq |A \cap (x_3 + H)|, \quad |B \cap (x_2 + H)| \geq |B \cap (x_4 + H)|,$$

hence

$$|A \cap (x_1 + H)|^{\frac{1}{2}}|B \cap (x_2 + H)|^{\frac{1}{2}} \geq \frac{1}{2K}\,|H|.$$

So far the proof of Theorem 4 is finished.

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